

# SERRIN'S OVERDETERMINED PROBLEM ON THE SPHERE

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ABSTRACT. We study Serrin's overdetermined boundary value problem

$$-\Delta_{S^N} u = 1 \quad \text{in } \Omega, \quad u = 0, \quad \partial_\eta u = \text{const} \quad \text{on } \partial\Omega$$

in subdomains  $\Omega$  of the round unit sphere  $S^N \subset \mathbb{R}^{N+1}$ , where  $\Delta_{S^N}$  denotes the Laplace-Beltrami operator on  $S^N$ . A subdomain  $\Omega$  of  $S^N$  is called a Serrin domain if it admits a solution of this overdetermined problem. In our main result, we construct Serrin domains in  $S^N$ ,  $N \geq 2$  which bifurcate from symmetric straight tubular neighborhoods of the equator. Our result provides the first example of Serrin domains in  $S^N$  which are not bounded by geodesic spheres.

## 1. INTRODUCTION AND MAIN RESULT

The present paper is concerned with an overdetermined boundary value problem for the equation

$$-\Delta_{S^N} u = 1 \quad \text{in } \Omega \tag{1.1}$$

in a subdomain  $\Omega$  of the unit sphere  $S^N \subset \mathbb{R}^{N+1}$ . More precisely, we are interested in domains  $\Omega \subset S^N$  of class  $C^2$  such that (1.1) is solvable subject to the boundary conditions

$$u = 0, \quad \partial_\eta u = \text{const} \quad \text{on } \partial\Omega. \tag{1.2}$$

Here  $\Delta_{S^N}$  is the Laplace-Beltrami operator on  $S^N$ , and  $\eta$  is the unit outer normal on  $\partial\Omega$ . A motivation to study this problem is given by classical and recent results for the corresponding problem in the euclidean setting, i.e. the problem of finding a smooth domain  $\Omega \subset \mathbb{R}^N$  such that the overdetermined problem

$$-\Delta u = 1 \quad \text{in } \Omega, \quad u = 0, \quad \partial_\nu u = \text{const} \quad \text{on } \partial\Omega \tag{1.3}$$

admits a solution. The additional Neumann boundary condition in (1.3) arises in many applications as a shape optimization problem for the underlying domain  $\Omega$ . For a detailed discussion of some applications e.g. in fluid dynamics and the linear theory of torsion, see [27, 29]. The most celebrated result for the overdetermined problem (1.3) has been obtained by Serrin in 1971. In his paper [27], Serrin proved that, among smooth bounded domains  $\Omega$ , (1.3) only admits a solution if  $\Omega$  is a ball. More precisely, in [27] the torsion equation  $-\Delta u = 1$  is considered as an important special case within a class of semilinear problems for which the same rigidity result is established. While

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Serrin's proof relies on the moving plane method which appeared for the first time in a PDE context in [27], Weinberger [31] found an alternative argument based on integral identities which yields the result in the special case of problem (1.3).

It is natural to ask if and in which sense Serrin's classification result carries over to the case of other ambient Riemannian manifolds  $(\mathcal{M}, g)$  in place of  $(\mathbb{R}^N, g_{eucl})$ . More precisely, letting  $\Delta_{\mathcal{M}}$  denote the Laplace-Beltrami operator on  $\mathcal{M}$ , we are interested in rigidity and non-rigidity results for *Serrin domains in  $\mathcal{M}$* , i.e., for subdomains  $\Omega \subset \mathcal{M}$  in which the overdetermined problem

$$-\Delta_{\mathcal{M}}u = 1 \quad \text{in } \Omega, \quad u = 0, \quad \partial_{\eta}u = \text{const} \quad \text{on } \partial\Omega \quad (1.4)$$

is solvable. As observed in [19], these domains arise in the context of Cheeger sets in a Riemannian framework. To explain this connection more precisely, we recall that the Cheeger constant of a Lipschitz subdomain  $\Omega \subset \mathcal{M}$  is given by

$$h(\Omega) := \inf_{A \subset \Omega} \frac{P(A)}{|A|}. \quad (1.5)$$

Here the infimum is taken over Lipschitz subsets  $A \subset \Omega$ , and  $|A|$  resp.  $P(A)$  denote the volume and perimeter of  $A$ , respectively (both with respect to the metric  $g$ ). If  $h(\Omega)$  is attained by  $\Omega$  itself,  $\Omega$  is usually called self-Cheeger, and it is called uniquely self-Cheeger if  $\Omega$  is the only subset which attains  $h(\Omega)$ . By means of the P-function method, it is shown in [19, Theorem 1.2] that every Serrin domain in a compact Riemannian manifold  $\mathcal{M}$  with nonnegative Ricci curvature is uniquely self-Cheeger. Cheeger constants play an important role in eigenvalue estimates on Riemannian manifolds (see [3]), whereas in the classical Euclidean case  $(\mathcal{M}, g) = (\mathbb{R}^N, g_{eucl})$  these notions have applications in the denoising problem in image processing, see e.g. [18, 21].

The result of Serrin was extended in [17] to subdomains  $\Omega$  of the round sphere  $\mathcal{M} = S^N$  which are contained in a hemisphere. More precisely, it is proved in [17] that any smooth Serrin domain  $\Omega$  contained in a hemisphere of  $S^N$  is a geodesic ball.

However, geodesic balls are not the only Serrin domains in  $S^N$ . Identifying  $S^{N-1}$  with the equator  $S^{N-1} \times \{0\} \subset S^N$ , it is easy to see that the symmetric straight tubular neighborhoods

$$\left\{ ((\cos \theta)\sigma, \sin \theta) : \sigma \in S^{N-1}, |\theta| < \lambda \right\} \subset S^N, \quad 0 < \lambda < \frac{\pi}{2} \quad (1.6)$$

of  $S^{N-1}$  are Serrin domains, see Section 2 below for details. We note that, as the geodesic balls, these domains are also bounded by geodesic spheres in  $S^N$ . It is therefore tempting to guess that this property is shared by any Serrin domain in  $S^N$ . In the present paper we show that this is not the case. More precisely, we construct Serrin domains in  $S^N$  of the form

$$D_{\phi} := \left\{ ((\cos \theta)\sigma, \sin \theta) : \sigma \in S^{N-1}, \theta \in \mathbb{R}, |\theta| < \phi(\sigma) \right\} \subset S^N, \quad (1.7)$$

where, in contrast to the sets given in (1.6),  $\phi : S^{N-1} \rightarrow \mathbb{R}$  is a nonconstant  $C^2$ -function with  $0 < \phi < \frac{\pi}{2}$ .

To state our main result, we need to introduce some notions. In the following, we call a function  $\phi : S^{N-1} \rightarrow \mathbb{R}$  *axially symmetric (with respect to the axis  $\mathbb{R}e_1$ )* if

$$\phi \equiv \phi \circ A \quad \text{for any } A \in O(N) \text{ with } Ae_1 = e_1. \quad (1.8)$$

Moreover, for a nonnegative integer  $j$ , we let  $Y_j$  be the unique  $L^2$ -normalized real-valued axially symmetric spherical harmonic of degree  $j$ . So  $Y_j$  is the restriction of the uniquely determined real-valued harmonic polynomial  $P_j$  of degree  $j$  to  $S^{N-1}$  which satisfies (1.8) and

$$\int_{S^{N-1}} P_j^2 d\sigma = \int_{S^{N-1}} Y_j^2 d\sigma = 1.$$

Here and in the following,  $d\sigma$  always denotes the volume element of  $S^{N-1}$ . The unique existence of  $Y_j$  is well known; a proof and some explicit formulas for  $Y_j$  are given e.g. in [14, Chapter 2]. Our main result is the following.

**Theorem 1.1.** *Let  $N \geq 2$  and  $\alpha \in (0, 1)$ . Then there exists a strictly decreasing sequence  $(\lambda_j)_{j \geq 2}$  with  $\lim_{j \rightarrow \infty} \lambda_j = 0$  and the following property:*

*For each  $j \geq 2$ , there exists  $\varepsilon_j > 0$  and a curve*

$$\begin{aligned} (-\varepsilon_j, \varepsilon_j) &\longrightarrow (0, \infty) \times C^{2,\alpha}(S^{N-1}) \\ s &\longmapsto (\lambda_j(s), \varphi_s^j) \end{aligned}$$

*with  $\varphi_0^j \equiv 0$ ,  $\lambda_j(0) = \lambda_j$  and such that for all  $s \in (-\varepsilon_j, \varepsilon_j)$ , letting  $\phi_s^j = \lambda_j(s) + \varphi_s^j$ , there exists a solution  $u \in C^{2,\alpha}(\overline{D_{\phi_s^j}})$  of the overdetermined problem*

$$\begin{cases} -\Delta_{S^N} u = 1 & \text{in } D_{\phi_s^j} \\ u = 0, \quad \partial_\nu u = \text{const} & \text{on } \partial D_{\phi_s^j}. \end{cases} \quad (1.9)$$

*Moreover, for every  $s \in (-\varepsilon_j, \varepsilon_j)$ , the function  $\varphi_s^j \in C^{2,\alpha}(S^{N-1})$  is axially symmetric. Furthermore, we have*

$$\varphi_s^j = s(Y_j + w_s^j) \quad \text{in } C^{2,\alpha}(S^{N-1}) \text{ for } s \in (-\varepsilon_j, \varepsilon_j),$$

*where  $s \mapsto w_s^j$  is a smooth map  $(-\varepsilon_j, \varepsilon_j) \rightarrow C^{2,\alpha}(S^{N-1})$  satisfying  $w_0^j \equiv 0$  and*

$$\int_{S^{N-1}} w_s^j(\sigma) Y_j(\sigma) d\sigma = 0 \quad \text{for } s \in (-\varepsilon_j, \varepsilon_j).$$

Theorem 1.1 supports the perception that the structure of the set of Serrin domains in a manifold  $(\mathcal{M}, g)$  is similar to the structure of closed embedded constant mean curvature hypersurfaces (CMC hypersurfaces in short) in  $\mathcal{M}$  up to a shift of the dimension. We recall that the rigidity result in [27] for Serrin domains in  $\mathbb{R}^N$  parallels an earlier important result by Alexandrov [1] stating that closed embedded CMC-hypersurfaces in  $\mathbb{R}^N$  are round spheres. The result in [1] is based on the moving plane method in geometric form which is also referred to as Alexandrov's reflection principle. By the same technique, Alexandrov proved in [2] that any closed embedded CMC-hypersurface contained in a hemisphere of  $S^N$  is a geodesic sphere. An explicit family of embedded

*CMC*-hypersurfaces in  $S^N$  with nonconstant principal curvatures has been found by Perdomo in [22] in the case  $N \geq 3$ . These hypersurfaces seem somewhat related to the Serrin domains given by Theorem 1.1, although they bound a tubular neighborhood of  $S^1$  and not of  $S^{N-1}$ . We also emphasize that Theorem 1.1 applies in the case of  $S^2$ , whereas *CMC*-hypersurfaces in  $S^2$  are obviously trivial, i.e., they are geodesic circles.

A similar dimensional shift is present in the case where  $\mathcal{M} = T^m \times \mathbb{R}^k$  with the flat metric, where  $T^m = \mathbb{R}^m / 2\pi\mathbb{Z}^m$  denotes the  $m$ -torus. In the recent paper [7], the authors have constructed bounded Serrin domains in  $\mathcal{M}$  with nonconstant principal curvatures for arbitrary  $m \geq 1$ ,  $k \geq 1$ . These domains clearly correspond to periodic unbounded domains in  $\mathbb{R}^m \times \mathbb{R}^k$ , and they should be seen as analogues of the classical one-parameter family of Delaunay hypersurfaces [5] in the case  $m = 1$ ,  $k = 2$ . Again we stress that there is no analogue in the two-dimensional case  $m = k = 1$ ; every bounded *CMC*-hypersurface in  $S^1 \times \mathbb{R}$  is clearly trivial, i.e., it is of the form  $S^1 \times \{t\}$  with  $t \in \mathbb{R}$ . On the other hand, the construction of Delaunay hypersurfaces has been generalized to higher dimensions in [16].

The richer structure of the set of Serrin domains in low dimensions can be understood as a consequence of the nonlocal nature of geometric optimization problems for overdetermined boundary value problems. As it will become clear in Section 2 below, these problems naturally lead to the study of (nonlocal) Dirichlet-to-Neumann maps.

This phenomenon has already been studied in detail for other classes of overdetermined boundary value problems involving different elliptic equations. Starting with the pioneering papers of Sicbaldi [28] and Hauswirth, Hélein and Pacard [15], the construction of nontrivial domains giving rise to solutions of overdetermined problems has been performed in many specific settings, see e.g. [6, 20, 24, 26]. Moreover, rigidity results for these domains were derived in [8–11, 23, 25, 30].

We note that most of the work so far has been devoted to the euclidean setting  $(\mathcal{M}, g) = (\mathbb{R}^N, g_{\text{eucl}})$ . In the very interesting recent paper [20], the more closely related case  $\mathcal{M} = S^{N-1} \times \mathbb{R}$  is considered for the equation  $-\Delta_{\mathcal{M}} u = \lambda u$ , but the domains considered there are unbounded with respect to  $\mathbb{R}$ -variable and thus very different from subdomains of  $S^N$ . In fact we are not aware of any other construction of nontrivial subdomains of the sphere  $S^N$  arising in the context of overdetermined boundary value problems.

Our approach for proving Theorem 1.1 relies on the Crandall-Rabinowitz Bifurcation Theorem [4, Theorem 1.7]. The main task is to reduce the problem to an operator equation in appropriate function spaces such that an associated family of linearized operators satisfy the spectral and Fredholm type transversality assumptions of this theorem. In particular, we need to derive precise information on the eigenvalues and eigenfunctions of related families of ODEs.

We explain the main steps while describing the organization of the paper. In Section 2, we consider domains of the form  $D_\phi$  given in (1.7) with  $\phi \in C^{2,\alpha}(S^{N-1})$ ,  $0 < \phi < \frac{\pi}{2}$  and transform the corresponding overdetermined problem to an equivalent overdetermined boundary value problem on a fixed domain with a  $\phi$ -dependent metric.

The solvability condition for this problem is then formulated as an operator equation in  $\phi$  of the form  $H(\phi) \equiv \text{const}$ , where  $H : C^{2,\alpha}(S^{N-1}) \rightarrow C^{1,\alpha}(S^{N-1})$  is a nonlinear operator of Dirichlet-to-Neumann type. We then compute its linearizations  $\mathbb{L}_\lambda := DH(\lambda)$  at constant functions  $\phi \equiv \lambda$ ,  $\lambda \in (0, \frac{\pi}{2})$  in the form of a classical Dirichlet-to-Neumann operator, see Proposition 2.4. In Section 3.1, we then study the eigenvalues  $\sigma_j(\lambda)$ ,  $j \in \mathbb{N} \cup \{0\}$  and corresponding eigenfunctions of  $\mathbb{L}_\lambda$  given by the spherical harmonics of degree  $j$ . By a separation of variables ansatz using spherical harmonics, we represent these eigenvalues via boundary derivatives of the solutions of an associated  $\lambda$ - and  $j$ -dependent family of second order ODEs, see (3.7) below. However, this representation is not too useful to derive detailed qualitative information. In Proposition 3.1, we therefore derive a different formula for the values  $\sigma_j(\lambda)$  which only involves a  $j$ -dependent family of first order ODEs *in the parameter*  $\lambda$ . This somewhat surprising new role of the parameter  $\lambda$  might be a consequence of a more general principle that we are unaware of so far. In Section 4, we use the information on the eigenvalues and eigenfunctions to deduce mapping properties of  $\mathbb{L}_\lambda$  in Sobolev spaces.

In Section 5, we then complete the proof of Theorem 1.1 via the Crandall-Rabinowitz Bifurcation Theorem. For this we need to restrict the operator  $H$  and its linearizations  $\mathbb{L}_\lambda$  to the subspace of axially symmetric functions in  $C^{2,\alpha}(S^{N-1})$ , because the intersections of this subspace with the eigenspaces of  $\mathbb{L}_\lambda$  are one-dimensional, as required in [4, Theorem 1.7]. We also note that, in order to ensure the transversality condition in [4, Theorem 1.7], we are led to show that  $\frac{d\sigma_j(\lambda)}{d\lambda} > 0$  for  $j \geq 2$  and  $\lambda \in (0, \frac{\pi}{2})$  with  $\sigma_j(\lambda) = 0$ , see Proposition 3.5 in Section 3.

Finally, we emphasize that Theorem 1.1 yields bifurcation of Serrin domains only at simple eigenvalues  $\sigma_j(\lambda_j)$  corresponding to axially symmetric spherical harmonics  $Y_j$  of higher degree  $j \geq 2$ . Such a bifurcation cannot happen for  $j = 1$ . Indeed, as we shall see in Proposition 3.1 below,  $\sigma_1(\lambda) < 0$ , for  $\lambda \in (0, \frac{\pi}{2})$ , while this eigenvalue corresponds to the (nonconstant) axially symmetric eigenfunction  $x \mapsto Y_1(x) = x_1$ . This contrasts our paper [7] devoted to  $\mathcal{M} = T^m \times \mathbb{R}^k$ , where bifurcation of Serrin's domain occurs at the every simple eigenvalue corresponding to a nonconstant eigenfunction.

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## 2. THE TRANSFORMED PROBLEM AND ITS LINEARIZATION

Consider the  $N$ -sphere  $S^N \subset \mathbb{R}^{N+1}$ . The open subset  $S^N \setminus \{\pm e_{N+1}\}$  is locally parameterized by

$$\Upsilon : S^{N-1} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow S^N, \quad (\sigma, \theta) \mapsto ((\cos \theta)\sigma, \sin \theta).$$

The standard metric of  $S^N$  in coordinates  $(\sigma, \theta)$  is given by

$$g_{(\sigma, \theta)} = \cos^2 \theta g_{S^{N-1}}(\sigma) + d\theta^2, \quad (2.1)$$

where  $g_{S^{N-1}}$  denotes the standard metric on  $S^{N-1}$ . In these coordinates, the corresponding Laplace-Beltrami operator of  $S^N$  writes as

$$\Delta_{g_{(\sigma, \theta)}} = \frac{1}{\cos^{N-1} \theta} \frac{\partial}{\partial \theta} \left( \cos^{N-1} \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\cos^2 \theta} \Delta_{S^{N-1}}. \quad (2.2)$$

We fix  $\alpha \in (0, 1)$  in the following. For  $j \in \mathbb{N} \cup \{0\}$ , we consider the Banach space  $C^{j, \alpha}(S^{N-1})$ , and we let

$$\mathcal{U} := \{\phi \in C^{2, \alpha}(S^{N-1}) : 0 < \phi < \frac{\pi}{2}\}.$$

For a function  $\phi \in \mathcal{U}$ , we define

$$\Omega_\phi := \left\{ (\sigma, \theta) \in S^{N-1} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) : |\theta| < \phi(\sigma) \right\}.$$

We note that  $\Upsilon(\Omega_\phi) = D_\phi$ , with  $D_\phi$  defined in (1.7). We are therefore led to consider the problem

$$\begin{cases} -\Delta_{g_{(\sigma, \theta)}} u = 1 & \text{in } \Omega_\phi, \\ u = 0 & \text{on } \partial\Omega_\phi. \end{cases} \quad (2.3)$$

We are looking for  $\phi \in \mathcal{U}$  such that the unique solution of (2.3) satisfies the additional boundary condition

$$\partial_{\mu_\phi} u = -c \quad \text{on} \quad \partial\Omega_\phi, \quad (2.4)$$

where  $\mu_\phi$  is the outer unit normal on  $\partial\Omega_\phi$  with respect to the metric  $g_{(\sigma, \theta)}$ , which is given by

$$\mu_\phi(\sigma, \theta) = \frac{\left(-\frac{\nabla \phi(\sigma)}{\cos \theta}, \frac{\theta}{|\theta|}\right)}{\sqrt{1 + |\nabla \phi(\sigma)|^2}} \in T_\sigma S^{N-1} \times \mathbb{R} \quad \text{for } (\sigma, \theta) \in \partial\Omega_\phi. \quad (2.5)$$

Here and in the following,  $\nabla \phi(\sigma) \in T_\sigma S^{N-1}$  denotes the gradient with respect to  $g_{S^{N-1}}$ .

In the following, we wish to pull back the problem (2.3) to a problem on a fixed domain. For this we note that every  $\phi \in \mathcal{U}$  gives rise to a  $C^{2, \alpha}$ -regular map

$$\Psi_\phi : S^{N-1} \times \mathbb{R} \rightarrow S^{N-1} \times \mathbb{R}, \quad \Psi_\phi(\sigma, t) = (\sigma, \phi(\sigma)t)$$

such that  $\Psi_\phi$  maps

$$\Omega := S^{N-1} \times (-1, 1)$$

diffeomorphically onto  $\Omega_\phi$ . Let the metric  $g_\phi$  be defined as the pull back of the metric  $g_{(\sigma, \theta)}$  under the map  $\Psi_\phi$ , so that  $\Psi_\phi : (\bar{\Omega}, g_\phi) \rightarrow (\bar{\Omega}_\phi, g_{(\sigma, \theta)})$  is an isometry. Then our problem is equivalent to the overdetermined problem consisting of

$$\begin{cases} -\Delta_{g_\phi} u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.6)$$

and the additional Neumann condition

$$\partial_{\nu_\phi} u = -c \quad \text{on} \quad \partial\Omega. \quad (2.7)$$

Here

$$\nu_\phi : \partial\Omega \rightarrow \mathbb{R}^{N+1}$$

is the unit outer normal at  $\partial\Omega \subset S^{N-1} \times \mathbb{R} \subset \mathbb{R}^{N+1}$  with respect to  $g_\phi$ . Since  $\Psi_\phi : (\overline{\Omega}, g_\phi) \rightarrow (\overline{\Omega}_\phi, g_{(\sigma, \theta)})$  is an isometry, we have

$$[d\Psi_\phi]\nu_\phi = \mu_\phi \circ \Psi_\phi \quad \text{on } \partial\Omega \quad (2.8)$$

with  $\mu_\phi$  is given in (2.5).

Here and in the following, we distinguish different types of derivatives in our notation. If  $f : O \rightarrow \mathbb{R}^\ell$  is a  $C^1$ -map defined on an open set  $O \subset S^{N-1} \times \mathbb{R}$ , we write  $df(\sigma, t) \in \mathcal{L}(T_\sigma S^{N-1} \times \mathbb{R}, \mathbb{R}^\ell)$  for its derivative at a point  $(\sigma, t) \in O$ . In contrast, we shall use the symbols  $D$  or  $D_\phi$  to denote functional derivatives. More precisely, if  $X, Y$  are infinite dimensional normed (function) spaces and  $F \in C^1(O, Y)$ , where  $O \subset X$  is open, we let  $DF(\phi)$  or  $D_\phi F(\phi) \in \mathcal{L}(X, Y)$  denote the Fréchet derivative of  $F$  at a function  $\phi \in O$ .

The following lemma is concerned with the well-posedness of problem (2.6).

**Lemma 2.1.** *For any  $\phi \in \mathcal{U}$ , there is a unique solution  $u_\phi \in C^{2,\alpha}(\overline{\Omega})$  of (2.6), and the map*

$$\mathcal{U} \rightarrow C^{2,\alpha}(\overline{\Omega}), \quad \phi \mapsto u_\phi \quad (2.9)$$

*is smooth. Moreover we have the following properties.*

- (i) *For any  $\phi \in \mathcal{U}$ , the functions  $u_\phi : \overline{\Omega} \rightarrow \mathbb{R}$  and  $\partial_{\nu_\phi} u_\phi : \partial\Omega \rightarrow \mathbb{R}$  are even in the variable  $t \in (-1, 1)$ .*
- (ii) *If  $\phi = \phi(\sigma, t)$  is axially symmetric in  $\sigma \in S^{N-1}$ , then the functions  $u_\phi : \overline{\Omega} \rightarrow \mathbb{R}$  and  $\partial_{\nu_\phi} u_\phi : \partial\Omega \rightarrow \mathbb{R}$  are axially symmetric in  $\sigma$  as well.*

**Proof.** Let

$$X := \{u \in C^{2,\alpha}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\} \quad \text{and} \quad Y = C^{0,\alpha}(\overline{\Omega}).$$

Moreover, let  $\mathcal{L}(X, Y)$  denote the space of bounded linear operators  $X \rightarrow Y$ , and let  $\mathcal{I}(X, Y) \subset \mathcal{L}(X, Y)$  denote the set of topological isomorphisms  $X \rightarrow Y$ . Since the metric coefficients of  $g_\phi$  are smooth functions of  $\phi$  and  $d\phi$ , it is easy to see that the map

$$m : \mathcal{U} \rightarrow \mathcal{L}(X, Y), \quad \phi \mapsto m(\phi) := -\Delta_{g_\phi}$$

is smooth. Moreover, for  $\phi \in \mathcal{U}$ , the definition of  $g_\phi$  implies that  $\Delta_{g_\phi}$  is an elliptic, coercive second order differential operator in divergence form with  $C^{1,\alpha}(\overline{\Omega})$ -coefficients. This readily implies, by the maximum principle and elliptic regularity, that  $m(\phi) \in \mathcal{I}(X, Y)$  for every  $\phi \in \mathcal{U}$ , and consequently the problem (2.6) has a unique solution  $u_\phi \in X$  for every  $\phi \in \mathcal{U}$ . We now recall that  $\mathcal{I}(X, Y) \subset \mathcal{L}(X, Y)$  is an open set and that the inversion

$$\text{inv} : \mathcal{I}(X, Y) \rightarrow \mathcal{I}(Y, X), \quad \text{inv}(A) = A^{-1}$$

is smooth. Since  $u_\phi = \text{inv}(m(\phi))1$ , the smoothness of the map in (2.9) follows.

Finally, properties (i) and (ii) follows from the uniqueness of  $u_\phi$ .  $\square$



By Lemma 2.1, condition (2.7) is equivalent to

$$[\partial_{\nu_\phi} u_\phi](\sigma, 1) = -c \quad \text{for } \sigma \in S^{N-1}. \quad (2.10)$$

It follows from (2.5) and (2.8) that the map

$$\mathcal{U} \rightarrow C^{1,\alpha}(\partial\Omega, \mathbb{R}^{N+1}), \quad \phi \mapsto \nu_\phi$$

is smooth, and thus we have a smooth map

$$H : \mathcal{U} \rightarrow C^{1,\alpha}(S^{N-1}), \quad H(\phi)(\sigma) = \partial_{\nu_\phi} u_\phi(\sigma, 1). \quad (2.11)$$

Moreover, (2.10) writes as

$$H(\phi) \equiv -c \quad \text{on } S^{N-1}. \quad (2.12)$$

In order to find solutions of the latter equation bifurcating from the trivial branch of solutions  $\phi \equiv \lambda$ ,  $\lambda > 0$ , we need to study the linearization of  $H$  at constant functions. In the next lemma, we collect the form of various quantities in the case  $\phi \equiv \lambda$  for later use.

**Lemma 2.2.** *In the case where  $\phi \equiv \lambda \in (0, \frac{\pi}{2})$  is a constant, we have*

$$\begin{aligned} \Omega_\lambda &= S^{N-1} \times ]-\lambda, \lambda[, \\ \mu_\lambda(\sigma, \theta) &= (0, \operatorname{sgn} \theta) \in T_\sigma S^{N-1} \times \mathbb{R} \quad \text{for } (\sigma, \theta) \in \partial\Omega_\lambda, \\ \Psi_\lambda(\sigma, t) &= (\sigma, \lambda t) \quad \text{for } (\sigma, t) \in S^{N-1} \times \mathbb{R}, \\ g_\lambda(\sigma, t) &= \lambda^2 dt^2 + \cos^2(\lambda t) g_{S^{N-1}}(\sigma) \quad \text{for } (\sigma, t) \in \overline{\Omega}, \\ \nu_\lambda(\sigma, t) &= (0, \frac{\operatorname{sgn}(t)}{\lambda}) \in T_\sigma S^{N-1} \times \mathbb{R} \quad \text{for } (\sigma, t) \in \partial\Omega \end{aligned}$$

and

$$\Delta_{g_\lambda} = \frac{1}{\lambda^2 \cos^{N-1}(\lambda t)} \partial_t \left( \cos^{N-1}(\lambda t) \partial_t \right) + \frac{1}{\cos^2(\lambda t)} \Delta_{S^{N-1}}, \quad (2.13)$$

where  $\Delta_{S^{N-1}}$  acts only on the  $\sigma$ -variable. Moreover, we have

$$u_\lambda(\sigma, t) = \tilde{u}(\lambda t) - \tilde{u}(\lambda) \quad \text{for } (\sigma, t) \in \overline{\Omega}, \quad (2.14)$$

where  $\tilde{u} : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$  is the even solution of the ODE

$$\frac{1}{\cos^{N-1} \theta} \frac{\partial}{\partial \theta} \left( \cos^{N-1} \theta \frac{\partial}{\partial \theta} \right) \tilde{u}(\theta) = -1, \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \quad (2.15)$$

which is unique up to an additive constant. Finally, we have

$$H(\lambda) = \tilde{u}'(\lambda). \quad (2.16)$$

**Proof.** The expressions for  $\Omega_\lambda$ ,  $\mu_\lambda$ ,  $\Psi_\lambda$ ,  $g_\lambda$  and  $\nu_\lambda$  are immediate. To prove (2.13), we note that

$$|g_\lambda(\sigma, t)| = \lambda^2 \cos^{2(N-1)}(\lambda t) |g_{S^{N-1}}|$$

and

$$g_\lambda^{-1}(\sigma, t) = \begin{pmatrix} \cos^{-2}(\lambda t) g_{S^{N-1}}^{-1}(\sigma) & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \quad \text{for } (\sigma, t) \in \overline{\Omega}, \quad (2.17)$$



so that

$$\begin{aligned}
\Delta_{g_\lambda} &= \frac{1}{\sqrt{|g_\lambda|}} \partial_i \left( \sqrt{|g_\lambda|} g_\lambda^{ij} \partial_j \right) = \frac{1}{\lambda \cos^{N-1}(\lambda t) \sqrt{|g_{S^{N-1}}|}} \\
&\quad \cdot \left[ \partial_t \left( \frac{\cos^{N-1}(\lambda t)}{\lambda} \sqrt{|g_{S^{N-1}}|} \partial_t \right) + \partial_i \left( \lambda \cos^{N-3}(\lambda t) \sqrt{|g_{S^{N-1}}|} g_{S^{N-1}}^{ij}(\sigma) \partial_j \right) \right] \\
&= \frac{1}{\lambda^2 \cos^{N-1}(\lambda t)} \partial_t \left( \cos^{N-1}(\lambda t) \partial_t \right) + \frac{1}{\cos^2(\lambda t) \sqrt{|g_{S^{N-1}}|}} \partial_i \left( \sqrt{|g_{S^{N-1}}|} g_{S^{N-1}}^{ij}(\sigma) \partial_j \right) \\
&= \frac{1}{\lambda^2 \cos^{N-1}(\lambda t)} \partial_t \left( \cos^{N-1}(\lambda t) \partial_t \right) + \frac{1}{\cos^2(\lambda t)} \Delta_{S^{N-1}},
\end{aligned}$$

as claimed in (2.13).

Next, let  $u_\lambda$  be given as in (2.14). Then, as a consequence of (2.21),  $u_\lambda$  solves the Dirichlet problem (2.6) with  $\phi \equiv \lambda$ , and this shows that the formula in (2.14) is correct. Finally, we have

$$H(\lambda) = \partial_{\nu_\lambda} u_\lambda = \tilde{u}'(\lambda)$$

as a consequence of (2.14), the fact that  $\partial_{\nu_\lambda} = \frac{\text{sgn}(t)}{\lambda} \partial_t$  and since  $\tilde{u}$  is even.  $\square$

We have the following estimate for the first derivative of the even solution  $\tilde{u}$  of (2.15).

**Lemma 2.3.** *We have*

$$-\tilde{u}'(\theta) \geq \theta \quad \text{for } \theta \in (0, \frac{\pi}{2}). \quad (2.18)$$

**Proof.** Since  $\tilde{u}$  is even, we have  $\tilde{u}'(0) = 0$  and therefore

$$\cos^{N-1} \theta \tilde{u}'(\theta) = - \int_0^\theta \cos^{N-1} \vartheta d\vartheta \quad \text{for } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

by (2.15). Consequently,

$$-\tilde{u}'(\theta) = \int_0^\theta \frac{\cos^{N-1} \vartheta}{\cos^{N-1} \theta} d\vartheta \geq \theta \quad \text{for } \theta \in (0, \frac{\pi}{2}).$$

$\square$

The following is the main result of this section.

**Proposition 2.4.** *Let  $H : \mathcal{U} \rightarrow C^{1,\alpha}(S^{N-1})$  be the smooth map defined in (2.11). At a constant function  $\lambda \in (0, \frac{\pi}{2})$ , the operator  $\mathbb{L}_\lambda := DH(\lambda) \in \mathcal{L}(C^{2,\alpha}(S^{N-1}), C^{1,\alpha}(S^{N-1}))$  is given by*

$$[\mathbb{L}_\lambda \omega](\sigma) = -\frac{\tilde{u}'(\lambda)}{\lambda} \partial_t \psi_{\omega,\lambda}(\sigma, 1) + \tilde{u}''(\lambda) \omega(\sigma) \quad \text{for } \omega \in C^{2,\alpha}(S^{N-1}), \sigma \in S^{N-1}, \quad (2.19)$$

where  $\psi_{\omega,\lambda} \in C^{2,\alpha}(\overline{\Omega})$  is the unique solution of the problem

$$\begin{cases} \Delta_{g_\lambda} \psi_{\omega,\lambda} = 0 & \text{in } \Omega, \\ \psi_{\omega,\lambda}(\sigma, t) = \omega(\sigma) & \text{for } (\sigma, t) \in \partial\Omega, \end{cases} \quad (2.20)$$

and

$$\Delta_{g_\lambda} = \frac{1}{\lambda^2 \cos^{N-1}(\lambda t)} \partial_t \left( \cos^{N-1}(\lambda t) \partial_t \right) + \frac{1}{\cos^2(\lambda t)} \Delta_{S^{N-1}}. \quad (2.21)$$

Here  $\Delta_{S^{N-1}}$  acts only on the  $\sigma$ -variable.

The remainder of this section is devoted to the proof of Proposition 2.4. To abbreviate the notation, we put

$$\tilde{\nu}_\phi(\omega) := [D_\phi \nu_\phi] \omega \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^{N+1}) \quad \text{for } \phi \in \mathcal{U}, \omega \in C^{2,\alpha}(S^{N-1}).$$

We start with the following simple observations.

**Lemma 2.5.**

(i) *The map*

$$\mathcal{U} \rightarrow C^{2,\alpha}(\overline{\Omega}, S^{N-1} \times \mathbb{R}) \subset C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^{N+1}), \quad \phi \mapsto \Psi_\phi$$

*is smooth. Moreover, for  $\omega \in C^{2,\alpha}(S^{N-1})$ , we have*

$$[D_\phi \Psi_\phi] \omega(\sigma, t) = (0, \omega(\sigma) t) \quad \text{for } (\sigma, t) \in \overline{\Omega}. \quad (2.22)$$

(ii) *Let*

$$m : \mathcal{U} \rightarrow C^{2,\alpha}(\overline{\Omega}), \quad \phi \mapsto m_\phi$$

*be a smooth map. Then the map*

$$\mathcal{M} : \mathcal{U} \rightarrow C^{1,\alpha}(\partial\Omega), \quad \mathcal{M}(\phi) = \partial_{\nu_\phi} m_\phi$$

*is smooth as well and satisfies*

$$D_\phi \mathcal{M}(\phi) \omega = \partial_{\tilde{\nu}_\phi(\omega)} m_\phi + \partial_{\nu_\phi} \left( [D_\phi m_\phi] \omega \right) \quad \text{for } \omega \in C^{2,\alpha}(S^{N-1}).$$

**Proof.** (i) follows immediately from the definition of  $\Psi_\phi$ .

(ii) For  $\phi \in \mathcal{U}$  and  $(\sigma, t) \in \partial\Omega$ , we have

$$\mathcal{M}(\phi)(\sigma, t) = \partial_{\nu_\phi} m_\phi(\sigma, t) = Dm_\phi(\sigma, t) \nu_\phi(\sigma, t).$$

Hence  $\mathcal{M}$  is smooth as a bilinear form composed with smooth functions, and  $D\mathcal{M}$  is given by

$$D_\phi \mathcal{M}(\phi) \omega = D[(D_\phi m_\phi) \omega] \nu_\phi + Dm_\phi \tilde{\nu}_\phi(\omega) = \partial_{\nu_\phi} \left( [D_\phi m_\phi] \omega \right) + \partial_{\tilde{\nu}_\phi(\omega)} m_\phi.$$

□

Next, we let  $\tilde{u}$  be fixed as in Lemma 2.3 and such that  $\tilde{u}(0) = 1$  (this normalization does not matter in the following). Moreover, for  $\phi \in \mathcal{U}$ , we define

$$u^\phi \in C^{2,\alpha}(\overline{\Omega}), \quad u^\phi(\sigma, t) = \tilde{u}(\phi(\sigma)t). \quad (2.23)$$

We then have

$$-\Delta_{g_\phi} u^\phi = 1 \quad \text{in } \Omega \quad (2.24)$$

since we can write  $u^\phi = \bar{u} \circ \Psi_\phi$  with the function

$$\bar{u} : S^{N-1} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}, \quad \bar{u}(\sigma, \theta) = \tilde{u}(\theta) \quad (2.25)$$

which solves

$$-\Delta_{g_{(\sigma, \theta)}} \bar{u} = 1 \quad \text{in } S^{N-1} \times (-\frac{\pi}{2}, \frac{\pi}{2}). \quad (2.26)$$

We need the following fact.

**Lemma 2.6.** *The map*

$$h : \mathcal{U} \rightarrow C^{1,\alpha}(S^{N-1}), \quad h(\phi) := \partial_{\nu_\phi} u^\phi(\cdot, 1) \in C^{1,\alpha}(S^{N-1})$$

*is smooth, and its derivative at a constant function  $\phi \equiv \lambda > 0$  satisfies*

$$[D_\phi|_{\phi=\lambda} h] \omega = \tilde{u}''(\lambda) \omega \quad \text{for } \sigma \in S^{N-1} \text{ and } \omega \in C^{2,\alpha}(S^{N-1}). \quad (2.27)$$

**Proof.** For  $\phi \in \mathcal{U}$ , recalling the definition of the outer normal  $\mu_\phi$  on  $\Omega_\phi$  given in (2.5), we define

$$\mu_\phi \in C^{1,\alpha}(S^{N-1}), \quad \mu_\phi(\sigma) = \mu_\phi(\Psi_\phi(\sigma, 1)) = \mu_\phi(\sigma, \phi(\sigma)).$$

By (2.8) and since  $\Psi_\phi : (\overline{\Omega}, g_\phi) \rightarrow (\overline{\Omega}_\phi, g_{(\sigma, \theta)})$  is an isometry, we have

$$\begin{aligned} h(\phi)(\sigma) &= \partial_{\nu_\phi} u^\phi(\sigma, 1) = \langle \nu_\phi(\sigma, 1), \nabla_{g_\phi} u^\phi(\sigma, 1) \rangle_{g_\phi} \\ &= \langle \mu_\phi(\sigma), \nabla_{g_{(\sigma, \theta)}} \bar{u}(\sigma, \phi(\sigma)) \rangle_{g_{(\sigma, \theta)}} \quad \text{for } \sigma \in S^{N-1}. \end{aligned} \quad (2.28)$$

Recalling that  $g_{(\sigma, \theta)} = d\theta^2 + \cos^2 \theta g_{S^{N-1}}(\sigma)$ , we get

$$\nabla_{g_{(\sigma, \theta)}} \bar{u}(\sigma, \theta) = (0, \tilde{u}'(\theta)) \in T_\sigma S^{N-1} \times \mathbb{R} \quad \text{for } (\sigma, \theta) \in S^{N-1} \times (-\frac{\pi}{2}, \frac{\pi}{2}),$$

so that (2.28) gives

$$h(\phi)(\sigma) = \langle \mu_\phi(\sigma), (0, \tilde{u}'(\phi(\sigma))) \rangle_{g_{(\sigma, \theta)}} \quad \text{for } \sigma \in S^{N-1}.$$

Consequently, for  $\omega \in C^{2,\alpha}(S^{N-1})$  we have

$$\begin{aligned} [D_\phi|_{\phi=\lambda} h(\phi)] \omega &= \langle [D_\phi|_{\phi=\lambda} \mu_\phi] \omega, (0, \tilde{u}'(\lambda)) \rangle_{g_{(\sigma, \theta)}} + \langle \mu_\lambda, (0, \tilde{u}''(\lambda) \omega) \rangle_{g_{(\sigma, \theta)}} \\ &= \tilde{u}'(\lambda) \langle [D_\phi|_{\phi=\lambda} \mu_\phi] \omega, \mu_\lambda \rangle_{g_{(\sigma, \theta)}} + \tilde{u}''(\lambda) \omega \quad \text{on } S^{N-1}. \end{aligned} \quad (2.29)$$

Here we used in the last step that

$$\mu_\lambda(\sigma) = (0, 1) \in T_\sigma S^{N-1} \times \mathbb{R} \quad \text{for all } \sigma \in S^{N-1}$$

by Lemma 2.2. Moreover, the first term in (2.29) vanishes since the map

$$\mathcal{U} \rightarrow C^{1,\alpha}(S^{N-1}), \quad \phi \mapsto |\mu_\phi|_{g_{(\sigma, \theta)}}^2 \equiv 1$$

is constant and thus

$$0 = [D_\phi |\boldsymbol{\mu}_\phi|_{g_{(\sigma,\theta)}}^2] \omega = 2 \langle [D_\phi \boldsymbol{\mu}_\phi] \omega, \boldsymbol{\mu}_\phi \rangle_{g_{(\sigma,\theta)}} \quad \text{on } S^{N-1}$$

for every  $\phi \in \mathcal{U}$ ,  $\omega \in C^{2,\alpha}(S^{N-1})$ . Thus (2.29) yields

$$[D_\phi|_{\phi=\lambda} h] \omega = \tilde{u}''(\lambda) \omega \quad \text{for } \omega \in C^{2,\alpha}(S^{N-1}),$$

as claimed.  $\square$

We are now in a position to complete the

**Proof of Proposition 2.4.** For  $\phi \in \mathcal{U}$ , we note that the function  $a_\phi := u_\phi - u^\phi \in C^{2,\alpha}(\overline{\Omega})$  satisfies

$$\begin{cases} -\Delta_{g_\phi} a_\phi = 0 & \text{in } \Omega \\ a_\phi = -u^\phi & \text{on } \partial\Omega. \end{cases} \quad (2.30)$$

Moreover, in the case where  $\phi \equiv \lambda > 0$ , we have

$$u^\lambda(\sigma, t) = \tilde{u}(\lambda t) \quad \text{and} \quad u_\lambda(\sigma, t) = \tilde{u}(\lambda t) - \tilde{u}(\lambda)$$

by (2.14), and hence

$$a_\lambda \equiv -\tilde{u}(\lambda) \equiv \text{const} \quad \text{on } \overline{\Omega}. \quad (2.31)$$

We now fix  $\omega \in C^{2,\alpha}(\overline{\Omega})$ , and we consider the smooth map  $T : \mathcal{U} \rightarrow C^{0,\alpha}(\overline{\Omega})$  given by

$$T(\phi) = \Delta_{g_\phi} a_\phi = g_\phi^{ij} \partial_{ij} a_\phi + \frac{1}{\sqrt{|g_\phi|}} \partial_i \left( \sqrt{|g_\phi|} g_\phi^{ij} \right) \partial_j a_\phi.$$

By (2.30) we have  $T \equiv 0$  on  $\mathcal{U}$ , and thus

$$0 = DT(\phi) \omega = \Delta_{g_\phi} [D_\phi a_\phi] \omega + h_{\phi,\omega}^{ij} \partial_{ij} a_\phi + \ell_\phi^j \partial_j a_\phi \quad \text{for } \phi \in \mathcal{U} \quad (2.32)$$

with

$$h_{\phi,\omega}^{ij} := [D_\phi g_\phi^{ij}] \omega \quad \text{and} \quad \ell_\phi^j := \left[ D_\phi \frac{1}{\sqrt{|g_\phi|}} \partial_i \left( \sqrt{|g_\phi|} g_\phi^{ij} \right) \right] \omega.$$

Evaluating (2.32) at  $\phi = \lambda$  and using (2.31), we find that

$$\Delta_{g_\lambda} \tau_{\omega,\lambda} = 0 \quad \text{in } C^{0,\alpha}(\overline{\Omega}) \quad \text{for } \tau_{\omega,\lambda} := [D_\phi|_{\phi=\lambda} a_\phi] \omega \in C^{2,\alpha}(\Omega). \quad (2.33)$$

Moreover, the boundary condition in (2.30) yields

$$a_\phi(\sigma, 1) = -u^\phi(\sigma, 1) = -\tilde{u}(\phi(\sigma))$$

and by differentiation at  $\phi \equiv \lambda$  we get

$$\tau_{\omega,\lambda}(\sigma, 1) = -\tilde{u}'(\lambda) \omega(\sigma) \quad \text{for } \sigma \in S^{N-1}. \quad (2.34)$$

By Lemma 2.5(ii) and (2.31) we also have that

$$[D_\phi|_{\phi=\lambda} \partial_{\nu_\phi} a_\phi(\cdot, 1)] \omega = \underbrace{\partial_{\tilde{\nu}_\phi(\omega)} a_\lambda(\cdot, 1)}_{=0} + \partial_{\nu_\lambda} \tau_{\omega,\lambda}(\cdot, 1) = \frac{1}{\lambda} \partial_t \tau_{\omega,\lambda}(\cdot, 1) \quad \text{on } S^{N-1}. \quad (2.35)$$

Using Lemma 2.6 and (2.35), we thus find that

$$\begin{aligned} [D_\phi|_{\phi=\lambda} H(\phi)]\omega &= [D_\phi|_{\phi=\lambda} \partial_{\nu_\phi} u_\phi(\cdot, 1)]\omega = [D_\phi|_{\phi=\lambda} \partial_{\nu_\phi} (a_\phi + u^\phi)(\cdot, 1)]\omega \\ &= \frac{1}{\lambda} \partial_t \tau_{\omega, \lambda}(\cdot, 1) + \tilde{u}''(\lambda)\omega \quad \text{on } S^{N-1}. \end{aligned}$$

Since  $\tilde{u}'(\lambda) < 0$  by Lemma 2.3, we may consider the function  $\psi_{\omega, \lambda} := -\frac{\tau_{\omega, \lambda}}{\tilde{u}'(\lambda)} \in C^{2, \alpha}(\overline{\Omega})$  which solves (2.20) as a consequence of (2.33) and (2.34), whereas

$$[D_\phi|_{\phi=\lambda} H(\phi)]\omega = -\frac{\tilde{u}'(\lambda)}{\lambda} \partial_t \psi_{\omega, \lambda}(\cdot, 1) + \tilde{u}''(\lambda)\omega \quad \text{on } S^{N-1},$$

as claimed.  $\square$

### 3. SPECTRAL PROPERTIES OF THE LINEARIZATION

In this section, we provide a detailed study of the spectral properties of the linearized operator  $\mathbb{L}_\lambda$  given in Proposition 2.4. For this we first recall that, if  $Y \in C^\infty(S^{N-1})$  is a spherical harmonic of degree  $j \in \mathbb{N} \cup \{0\}$ , then we have

$$-\Delta_{S^{N-1}} Y = \gamma_j Y \quad \text{with} \quad \gamma_j := j(N-2+j). \quad (3.1)$$

This property allows to determine the eigenvalues of  $\mathbb{L}_\lambda$ , as we shall see in the following proposition.

**Proposition 3.1.** *Let  $\lambda \in (0, \frac{\pi}{2})$ , and let  $Y \in C^\infty(S^{N-1})$  be a spherical harmonic of degree  $j \in \mathbb{N} \cup \{0\}$ . Then  $Y$  is an eigenfunction of  $\mathbb{L}_\lambda$ . More precisely, we have*

$$\mathbb{L}_\lambda Y = \sigma_j(\lambda) Y \quad \text{on } S^{N-1}, \quad (3.2)$$

with

$$\sigma_j(\lambda) := \tilde{u}'(\lambda) \left( (N-1) \tan \lambda - f_j(\lambda) \right) - 1, \quad (3.3)$$

where  $\tilde{u}$  is the solution to (2.15) and  $f_j : [0, \frac{\pi}{2}) \rightarrow \mathbb{R}$  is the unique solution of the initial value problem

$$\begin{cases} f_j'(\lambda) = -f_j^2(\lambda) + (N-1) \tan \lambda f_j(\lambda) + \frac{\gamma_j}{\cos^2 \lambda}, & \lambda \in (0, \frac{\pi}{2}) \\ f_j(0) = 0. \end{cases} \quad (3.4)$$

Moreover, we have

$$f_0(\lambda) = 0, \quad f_1(\lambda) = (N-1) \tan \lambda \quad \text{for } \lambda \in [0, \frac{\pi}{2}) \quad (3.5)$$

and

$$c_j \tan \lambda \leq f_j(\lambda) \leq N \frac{\sqrt{\gamma_j}}{\cos \lambda} \quad \text{for } \lambda \in [0, \frac{\pi}{2}) \text{ if } j \geq 2, \quad (3.6)$$

where  $c_j := \frac{N-2+\sqrt{(N-2)^2+4\gamma_j}}{2} > N-1$ .

**Proof.** For  $\omega = Y$ , the unique solution  $\psi_{\omega,\lambda}$  of (2.20) is given by  $\psi_{\omega,\lambda}(\sigma, t) = b_{j,\lambda}(t)Y(\sigma)$ , where the function  $b_{j,\lambda} : [0, 1] \rightarrow \mathbb{R}$  is the unique solution to

$$\begin{cases} b''_{j,\lambda}(t) - (N-1)\lambda \tan(\lambda t)b'_{j,\lambda}(t) - \frac{\lambda^2 \gamma_j}{\cos^2(\lambda t)}b_{j,\lambda}(t) = 0 \\ b'_{j,\lambda}(0) = 0, \quad b_{j,\lambda}(1) = 1. \end{cases} \quad (3.7)$$

We conclude that (3.2) holds with

$$\sigma_j(\lambda) = -\frac{\tilde{u}'(\lambda)}{\lambda}b'_{j,\lambda}(1) + \tilde{u}''(\lambda) = \tilde{u}'(\lambda)\left((N-1)\tan \lambda - \frac{b'_{j,\lambda}(1)}{\lambda}\right) - 1,$$

where we used in the last step that

$$\tilde{u}''(\lambda) = -1 + (N-1)\tan \lambda \tilde{u}'(\lambda) \quad \text{for } \lambda \in [0, \frac{\pi}{2}) \quad (3.8)$$

by (2.15). Next we prove that (3.4) admits a unique solution  $f_j$  on  $[0, \frac{\pi}{2})$ , and that

$$f_j(\lambda) = \frac{b'_{j,\lambda}(1)}{\lambda} \quad \text{for } \lambda \in (0, \frac{\pi}{2}). \quad (3.9)$$

For this, we make a change of variable to extract the  $\lambda$ -dependence of  $b_{j,\lambda}(1)$ . Let  $B_j : (-1, 1) \rightarrow \mathbb{R}$  be the unique solution of the linear initial value problem

$$\begin{cases} B''_j - \frac{Nz}{1-z^2}B'_j - \frac{\gamma_j}{(1-z^2)^2}B_j = 0, & z \in (-1, 1) \\ B_j(0) = 1, & B'_j(0) = 0. \end{cases} \quad (3.10)$$

We first note that

$$B_j \text{ is positive and satisfies } B'_j > 0 \text{ on } (0, 1). \quad (3.11)$$

Indeed, by the differential equation in (3.10), any point  $z \in (-1, 1)$  with  $B'_j(z) = 0$  is a strict local minimum if  $B_j(z) > 0$  and a strict local maximum if  $B_j(z) < 0$ . Combining this property with the initial conditions in (3.10), we obtain (3.11).

We now have the identity

$$b_{j,\lambda}(t) = \frac{B_j(\sin(\lambda t))}{B_j(\sin \lambda)} \quad \text{for } \lambda \in (0, \frac{\pi}{2}), \quad t \in [-1, 1], \quad (3.12)$$

which can be easily checked by comparing (3.7) and (3.10). Moreover, the unique solution  $f_j : [0, \frac{\pi}{2}) \rightarrow \mathbb{R}$  of (3.4) is given by

$$f_j(\lambda) := \cos \lambda \frac{B'_j(\sin \lambda)}{B_j(\sin \lambda)}. \quad (3.13)$$

Indeed, differentiation yields

$$\begin{aligned} f'_j(\lambda) &= -\sin \lambda \frac{B'_j(\sin \lambda)}{B_j(\sin \lambda)} + \cos^2 \lambda \frac{B''_j(\sin \lambda)B_j(\sin \lambda) - [B'_j(\sin \lambda)]^2}{B_j^2(\sin \lambda)} \\ &= -\tan \lambda f_j(\lambda) + \cos^2 \lambda \frac{B''_j(\sin \lambda)}{B_j(\sin \lambda)} - f_j^2(\lambda), \end{aligned} \quad (3.14)$$

whereas

$$\cos^2 \lambda B''_j(\sin \lambda) = N \sin \lambda B'_j(\sin \lambda) + \frac{\gamma_j B_j(\sin \lambda)}{\cos^2 \lambda}$$

by (3.10). Inserting this in (3.14) gives (3.4), as claimed. Moreover, by (3.12) we have

$$b'_{j,\lambda}(1) = \frac{\lambda \cos \lambda B'_j(\sin \lambda)}{B_j(\sin \lambda)} = \lambda f_j(\lambda) \quad \text{for } \lambda \in (0, \frac{\pi}{2}), \quad (3.15)$$

as claimed in (3.9).

Next we note that (3.5) is an obvious consequence of (3.4). To complete the proof, it thus remains to prove (3.6). So fix  $j \geq 2$ . Then  $\gamma_j > N-1$ , and thus the unique positive root  $c = c_j$  of the equation  $c^2 - (N-2)c - \gamma_j = 0$  satisfies  $c_j = \frac{N-2 + \sqrt{(N-2)^2 + 4\gamma_j}}{2} > N-1$ . Moreover, the function

$$\lambda \mapsto g_j(\lambda) = c_j \tan \lambda$$

satisfies  $g(0) = 0$  and

$$\begin{aligned} g'_j(\lambda) + g_j^2(\lambda) - (N-1) \tan \lambda g_j(\lambda) - \frac{\gamma_j}{\cos^2 \lambda} &= \frac{c_j + \sin^2 \lambda [c_j^2 - (N-1)c_j] - \gamma_j}{\cos^2 \lambda} \\ &\leq \frac{c_j^2 - (N-2)c_j - \gamma_j}{\cos^2 \lambda} = 0 \end{aligned}$$

for  $\lambda \in (0, \frac{\pi}{2})$ , so that the function  $h_j = f_j - g_j$  satisfies  $h_j(0) = 0$  and

$$h'_j(\lambda) \geq g_j^2(\lambda) - f_j^2(\lambda) + (N-1) \tan \lambda h_j(\lambda) = \left( (N-1) \tan \lambda - f_j(\lambda) - g_j(\lambda) \right) h_j(\lambda).$$

By Gronwall's inequality, this implies that  $h_j \geq 0$  on  $[0, \frac{\pi}{2})$ , and thus

$$f_j(\lambda) \geq g_j(\lambda) = c_j \tan \lambda \quad \text{for } \lambda \in [0, \frac{\pi}{2}),$$

which gives the lower estimate in (3.6).

To see the upper bound in (3.6), we argue by contradiction. So we consider the function  $\lambda \mapsto F_j(\lambda) := f_j(\lambda) - N \frac{\sqrt{\gamma_j}}{\cos \lambda}$  and assume that the set

$$M := \{ \lambda \in (0, \frac{\pi}{2}) : F_j(\lambda) \geq 0 \}$$



is nonempty. Since  $F_j(0) = -N\sqrt{\gamma_j} < 0$  and  $F_j$  is continuous, it then follows that there exists  $\lambda^* := \min M \in (0, \frac{\pi}{2})$ , whereas  $F_j(\lambda^*) = 0$ , i.e.,  $f_j(\lambda^*) = N\frac{\sqrt{\gamma_j}}{\cos \lambda^*}$  and

$$\begin{aligned} 0 &\leq F'_j(\lambda^*) = -f_j^2(\lambda^*) + (N-1)\tan \lambda^* f_j(\lambda^*) + \frac{\gamma_j}{\cos^2 \lambda^*} - N\sqrt{\gamma_j} \frac{\sin \lambda^*}{\cos^2 \lambda^*} \\ &\leq -N^2 \frac{\gamma_j}{\cos^2 \lambda^*} + N(N-1)\tan \lambda^* \frac{\sqrt{\gamma_j}}{\cos \lambda^*} + \frac{\gamma_j}{\cos^2 \lambda^*} \\ &\leq \frac{1}{\cos^2 \lambda^*} \left( (1-N^2)\gamma_j + N(N-1)\sqrt{\gamma_j} \right) \leq \frac{N^2-1}{\cos^2 \lambda^*} (\sqrt{\gamma_j} - \gamma_j) < 0. \end{aligned}$$

This is a contradiction, and thus the upper bound in (3.6) follows. The proof is finished.  $\square$

Proposition 3.1 allows us to easily derive important properties of the shape and asymptotics of the eigenvalue curves  $\lambda \mapsto \sigma_j(\lambda)$ . The following lemmas will be crucial in order to apply the Crandall-Rabinowitz theorem.

**Lemma 3.2.** *For every  $\lambda \in (0, \frac{\pi}{2})$  we have*

$$0 < \liminf_{j \rightarrow \infty} \frac{\sigma_j(\lambda)}{j} \leq \limsup_{j \rightarrow \infty} \frac{\sigma_j(\lambda)}{j} < \infty.$$

**Proof.** Since  $\tilde{u}'(\lambda) < 0$  for  $\lambda \in (0, \frac{\pi}{2})$  by (2.18), the claim follows from (3.3) and (3.6).

**Lemma 3.3.** *For every  $\lambda \in (0, \frac{\pi}{2})$  and  $i, j \in \mathbb{N} \cup \{0\}$ ,  $i < j$  we have  $\sigma_j(\lambda) > \sigma_i(\lambda)$ .*

**Proof.** This follows readily from (3.3) and the fact that

$$f_j(\lambda) > f_i(\lambda) \quad \text{for } i, j \in \mathbb{N} \cup \{0\}, i < j. \quad (3.16)$$

To see (3.16), note that  $h := f_j - f_i$  solves the initial value problem

$$\begin{cases} h'(\lambda) = a(\lambda)h(\lambda) + \frac{\gamma_j - \gamma_i}{\cos^2 \lambda}, & \lambda \in (0, \frac{\pi}{2}), \\ h(0) = 0, \end{cases} \quad (3.17)$$

with  $a(\lambda) = (N-1)\tan \lambda - f_j(\lambda) - f_i(\lambda)$ . Since  $\gamma_j - \gamma_i > 0$ , this implies that  $h(\lambda) > 0$  for  $\lambda \in (0, \frac{\pi}{2})$ , and thus (3.16) holds.  $\square$

**Lemma 3.4.** *The eigenvalue curves  $\lambda \mapsto \sigma_j(\lambda)$  have the following asymptotics.*

- (i)  $\lim_{\lambda \rightarrow 0} \sigma_j(\lambda) = -1$  for every  $j \geq 0$ .
- (ii)  $\lim_{\lambda \rightarrow \frac{\pi}{2}^-} \sigma_j(\lambda) = \infty$  for every  $j \geq 2$ .

**Proof.** Claim (i) follows immediately from (3.3) and (3.4).

Moreover, for  $j \geq 2$ , (2.18), (3.3) and (3.6) yield that

$$\sigma_j(\lambda) \geq (c_j - N - 1)\lambda \tan \lambda \rightarrow \infty \quad \text{as } \lambda \rightarrow \frac{\pi}{2}^-,$$

as claimed.  $\square$

**Proposition 3.5.** *For every  $j \geq 2$ , there exists a unique  $\lambda_j \in (0, \frac{\pi}{2})$  such that  $\sigma_j(\lambda_j) = 0$ . Moreover, we have*

- (i)  $\sigma'_j(\lambda_j) > 0$  for  $j \geq 2$ ;
- (ii)  $\lambda_j < \lambda_i$  for  $2 \leq i < j$ ;
- (iii)  $\lim_{j \rightarrow \infty} \lambda_j = 0$ .

**Proof.** Let  $j \geq 2$ , and suppose that  $\lambda_* \in (0, \frac{\pi}{2})$  satisfies  $\sigma_j(\lambda_*) = 0$ . We first claim that

$$\sigma'_j(\lambda_*) > 0. \quad (3.18)$$

Indeed, differentiating (3.3) at  $\lambda = \lambda_*$  gives

$$\sigma'_j(\lambda_*) = \tilde{u}''(\lambda_*) \left( (N-1) \tan \lambda_* - f_j(\lambda_*) \right) + \tilde{u}'(\lambda_*) \left( \frac{N-1}{\cos^2 \lambda_*} - f'_j(\lambda_*) \right). \quad (3.19)$$

Moreover, (3.3) gives

$$0 = \sigma_j(\lambda_*) = (N-1) \tan \lambda_* \tilde{u}'(\lambda_*) - f_j(\lambda_*) \tilde{u}'(\lambda_*) - 1,$$

and thus, by (3.8),

$$\tilde{u}''(\lambda_*) = -1 + (N-1) \tan \lambda_* \tilde{u}'(\lambda_*) = f_j(\lambda_*) \tilde{u}'(\lambda_*).$$

Inserting this in (3.19) and using (3.4), we deduce that

$$\sigma'_j(\lambda_*) = \tilde{u}'(\lambda_*) \left( (N-1) \tan \lambda_* f_j(\lambda_*) - f_j^2(\lambda_*) + \frac{N-1}{\cos^2 \lambda_*} - f'_j(\lambda_*) \right) = \tilde{u}'(\lambda_*) \frac{N-1-\gamma_j}{\cos^2 \lambda_*}.$$

Since  $\tilde{u}' < 0$  on  $(0, \frac{\pi}{2})$  and  $\gamma_j > N-1$  for  $j \geq 2$ , (3.18) follows.

By combining (3.18), Lemma 3.4 and the intermediate value theorem, it then follows that for every  $j \geq 2$  there exists a unique  $\lambda_j \in (0, \frac{\pi}{2})$  such that  $\sigma_j(\lambda_j) = 0$ , and that (i) holds. Claim (ii) then follows immediately from Lemma 3.3. Moreover, by (2.18), (3.3) and (3.6) we have

$$1 = \sigma(\lambda_j) + 1 = \tilde{u}'(\lambda_j) \left( (N-1) \tan \lambda_j - f_j(\lambda_j) \right) \geq (c_j - N + 1) \lambda_j \tan \lambda_j$$

for  $j \geq 2$ . Since  $c_j \rightarrow \infty$ , it follows that  $\lambda_j \rightarrow 0$  as  $j \rightarrow \infty$ , as claimed in (iii).  $\square$

#### 4. MAPPING PROPERTIES IN SOBOLEV SPACES

In the following, we consider the Sobolev spaces

$$H^k := H^k(S^{N-1}), \quad k \in \mathbb{N} \cup \{0\}, \quad (4.1)$$

and as usual we put  $L^2 := H^0$ . Note that  $L^2$  is a Hilbert space with scalar product

$$(u, v) \mapsto \langle u, v \rangle_{L^2} := \int_{S^{N-1}} uv \, d\sigma \quad \text{for } u, v \in L^2.$$

In the following, for  $\ell \in \mathbb{N} \cup \{0\}$ , we let

$$V_\ell \subset \bigcap_{k \in \mathbb{N}} H^k \quad (4.2)$$

denote the space of spherical harmonics of degree  $\ell$ . Moreover, we let  $P_\ell : L^2 \rightarrow L^2$  denote the  $\langle \cdot, \cdot \rangle_{L^2}$ -orthogonal projections on  $V_\ell$  and

$$W_\ell^k := \{v \in H^k : P_\ell v = 0\} \subset H^k, \quad (4.3)$$

the  $\langle \cdot, \cdot \rangle_{L^2}$ -orthogonal complements of  $V_\ell$  in  $H^k$ . We may then define a scalar product on  $H^k$  by setting

$$(u, v) \mapsto \langle u, v \rangle_{H^k} := \sum_{\ell \in (\mathbb{N} \cup \{0\})} (1 + \ell^2)^k \langle P_\ell u, P_\ell v \rangle_{L^2} \quad \text{for } u, v \in H^k, \quad (4.4)$$

and the induced norm of this scalar product is equivalent to the standard norm on  $H^k$ .

**Proposition 4.1.** *For fixed  $\lambda \in (0, \frac{\pi}{2})$ , the linear map  $\mathbb{L}_\lambda$  defined in (2.19) extends to a continuous linear map*

$$\mathbb{L}_\lambda : H^2 \rightarrow H^1, \quad \mathbb{L}_\lambda v = \sum_{\ell \in \mathbb{N} \cup \{0\}} \sigma_\ell(\lambda) P_\ell v.$$

Moreover, for any  $\ell \in \mathbb{N} \cup \{0\}$ , the operator

$$\mathbb{L}_\lambda - \sigma_\ell(\lambda) \text{id} : W_\ell^2 \rightarrow W_\ell^1 \quad \text{is an isomorphism.}$$

**Proof.** This follows from Proposition 3.1, Lemma 3.2, Lemma 3.3 and the remarks above.  $\square$

**Remark 4.2.** *The extension  $\mathbb{L}_\lambda : H^2 \rightarrow H^1$  given in Proposition 4.1 can be characterized as follows. Given  $\omega \in H^2$ , let  $\psi \in W^{1,2}(\Omega)$  be the unique weak solution of the problem*

$$\begin{cases} \Delta_{g_\lambda} \psi = 0 & \text{in } \Omega, \\ \psi(\sigma, t) = \omega(\sigma) & \text{for } (z, t) \in \partial\Omega, \end{cases} \quad (4.5)$$

which is axially symmetric in the  $\sigma$ -variable, where  $\Delta_{g_\lambda}$  is given by (2.21). By standard elliptic regularity theory, we then have  $\psi \in W^{2,2}(\Omega)$ , and  $\mathbb{L}_\lambda \omega$  is given by

$$[\mathbb{L}_\lambda \omega](\sigma) = -\frac{\tilde{u}'(\lambda)}{\lambda} \partial_t \psi(\sigma, 1) + \tilde{u}''(\lambda) \omega(\sigma) \quad \text{for a.e. } \sigma \in S^{N-1},$$

where  $\partial_t \psi$  is considered in the sense of traces. This can be easily seen by approximating  $\omega$  in  $H^2$  with functions in  $C^{2,\alpha}(\Omega)$  and using standard elliptic estimates.

## 5. PROOF OF THEOREM 1.1

In the following, we consider the spaces

$$\begin{aligned} \mathcal{X} &:= \{\varphi \in C^{2,\alpha}(S^{N-1}) : \varphi \text{ axially symmetric}\}, \\ \mathcal{Y} &:= \{\varphi \in C^{1,\alpha}(S^{N-1}) : \varphi \text{ axially symmetric}\}. \end{aligned}$$

Here axial symmetry is defined with respect to the axis  $\mathbb{R}e_1$  as in the introduction, see (1.8). We also consider the nonlinear operator  $H$  defined in (2.11), and we note that  $H$  maps  $\mathcal{U} \cap \mathcal{X}$  into  $\mathcal{Y}$  by Lemma 2.1(iii). Consider the open set

$$\mathcal{O} := \{(\lambda, \varphi) \in (0, \frac{\pi}{2}) \times \mathcal{X} : -\lambda < \varphi < -\lambda + \frac{\pi}{2}\} \subset \mathbb{R} \times \mathcal{X}.$$

In view of (2.11) and (2.16), the proof of Theorem 1.1 will be completed by applying the Crandall-Rabinowitz bifurcation theorem to the smooth nonlinear operator

$$G : \mathcal{O} \subset \mathbb{R} \times \mathcal{X} \rightarrow \mathcal{Y}, \quad G(\lambda, \varphi) = H(\lambda + \varphi) - H(\lambda) = H(\lambda + \varphi) - \tilde{u}'(\lambda). \quad (5.1)$$

By (2.16), we have  $G(\lambda, 0) = 0$  in  $\mathcal{Y}$  for all  $\lambda \in (0, \frac{\pi}{2})$ . Moreover,

$$D_\varphi G(\lambda, 0) = DH(\lambda)|_{\mathcal{X}} = \mathbb{L}_\lambda|_{\mathcal{X}} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}). \quad (5.2)$$

We have the following

**Proposition 5.1.** *Consider  $\lambda_j$ , with  $j \geq 2$ , as given in Proposition 3.5. For  $j \geq 2$ , the linear operator*

$$\mathbb{L}_j := \mathbb{L}_{\lambda_j}|_{\mathcal{X}} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$$

*has the following properties.*

- (i) *The kernel  $N(\mathbb{L}_j)$  of  $\mathbb{L}_j$  is spanned by  $Y_j$ , the unique  $L^2$ -normalized real-valued axially symmetric spherical harmonic of degree  $j$  defined in the introduction.*
- (ii) *The range of  $\mathbb{L}_j$  is given by*

$$R(\mathbb{L}_j) = \left\{ v \in \mathcal{Y} : \int_{S^{N-1}} v Y_j d\sigma = 0 \right\}.$$

Moreover,

$$\partial_\lambda \Big|_{\lambda=\lambda_j} \mathbb{L}_\lambda Y_j \notin R(\mathbb{L}_j). \quad (5.3)$$

**Proof.** By definition we have  $\sigma_j(\lambda_j) = 0$ , which by Proposition 3.1 is equivalent to  $\mathbb{L}_j Y_j = 0$ . We consider the subspaces

$$\begin{aligned} \mathcal{X}_j &:= \left\{ v \in \mathcal{X} : \int_{S^{N-1}} v Y_j d\sigma = 0 \right\} \subset \mathcal{X}, \\ \mathcal{Y}_j &:= \left\{ v \in \mathcal{Y} : \int_{S^{N-1}} v Y_j d\sigma = 0 \right\} \subset \mathcal{Y}. \end{aligned} \quad (5.4)$$

To show properties (i) and (ii), it clearly suffices to prove that

$$\mathbb{L}_j \text{ defines an isomorphism between } \mathcal{X}_j \text{ and } \mathcal{Y}_j. \quad (5.5)$$

To prove (5.5), we need to introduce further spaces. We recall the definition of the Sobolev space  $H^k$  in (4.1) and put

$$H_{ax}^k := \{v \in H^k : v \text{ axially symmetric}\}, \quad k \in \mathbb{N} \cup \{0\}, \quad (5.6)$$

noting that  $\mathcal{X} = H_{ax}^2 \cap C^{2,\alpha}(S^{N-1})$  and  $\mathcal{Y} = H_{ax}^1 \cap C^{1,\alpha}(S^{N-1})$ . Proposition 4.1 implies that  $\mathbb{L}_j$  defines a continuous linear operator

$$\mathbb{L}_j : H_{ax}^2 \rightarrow H_{ax}^1, \quad \mathbb{L}_j v = \sum_{\ell \in \mathbb{N} \cup \{0\}} \sigma_\ell(\lambda_j) P_\ell v. \quad (5.7)$$

Next we put

$$\tilde{V}_j^k := H_{ax}^k \cap V_j, \quad \tilde{W}_j^k := H_{ax}^k \cap W_j^k \subset H_{ax}^k \quad \text{for } k = 1, 2,$$

where the spaces  $V_j$  resp.  $W_j^k$  are defined in (4.2) and (4.3), respectively. We note that  $\tilde{V}_j^k$  is one-dimensional and spanned by the function  $Y_j$  given in (i). We then deduce from Proposition 4.1 and the property  $\sigma_j(\lambda_j) = 0$  that

$$\mathbb{L}_j \text{ defines an isomorphism } \tilde{W}_j^2 \rightarrow \tilde{W}_j^1. \quad (5.8)$$

Moreover, since  $\mathcal{X}_j = \tilde{W}_j^2 \cap \mathcal{X}$  and  $\mathcal{Y}_j = \tilde{W}_j^1 \cap \mathcal{Y}$ , we see that  $\mathbb{L}_j : \mathcal{X}_j \rightarrow \mathcal{Y}_j$  is well defined and injective. To establish surjectivity, we let  $f \in \mathcal{Y}_j$ . By (5.8), there exists  $\omega \in \tilde{W}_j^2 \subset H^2$  such that  $\mathbb{L}_j \omega = f$ . As noted in Remark 4.2, we then have

$$-\frac{\tilde{u}'(\lambda_j)}{\lambda_j} \partial_\nu \psi(\sigma, \pm 1) + \tilde{u}''(\lambda_j) \omega(\sigma) = f(\sigma) \quad \text{for a.e. } \sigma \in S^{N-1}, \quad (5.9)$$

where  $\psi \in W_{loc}^{2,2}(\Omega)$  is the unique solution of (4.5) with  $\lambda = \lambda_j$ , recalling that  $\Omega = S^{N-1} \times (-1, 1)$ . We claim that

$$\psi \in C^{2,\alpha}(\overline{\Omega}). \quad (5.10)$$

Indeed, this follows from [13, Theorem 6.3.2.1] once we have shown that

$$\psi \in W^{2,p}(\Omega) \quad \text{for some } p > N. \quad (5.11)$$

To see this, we show by induction that

$$\psi \in W^{2,p_k}(\Omega) \quad (5.12)$$

for a sequence of numbers  $p_k \in [2, \infty)$  satisfying  $p_0 = 2$  and  $p_{k+1} \geq \frac{N-1}{N-2} p_k$  for  $k \geq 0$ . We already know that (5.12) holds for  $p_0 = 2$ . So let us assume that (5.12) holds for some  $p_k \geq 2$ . We distinguish two cases.

If  $p_k < N$ , then the trace theorem implies that

$$\psi|_{\partial\Omega} \in W^{1,p_{k+1}}(\partial\Omega) \quad \text{with } p_{k+1} := \left(\frac{N-1}{N-p_k}\right)p_k \geq \frac{N-1}{N-2} p_k,$$

so that  $\omega \in W^{1,p_{k+1}}(S^{N-1})$ . Since also  $f \in C^{1,\alpha}(S^{N-1}) \subset W^{1,p_{k+1}}(S^{N-1})$  and, by (5.9),

$$-\frac{\tilde{u}'(\lambda_j)}{\lambda_j} \partial_\nu \psi(\sigma, \pm 1) + \psi(\sigma, \pm 1) = g(\sigma) \quad \text{for a.e. } \sigma \in S^{N-1},$$

with

$$g = (1 - \tilde{u}''(\lambda_j))\omega + f \in W^{1,p_{k+1}}(S^{N-1}),$$

we may deduce from [13, Theorem 2.4.2.6] that  $\psi \in W^{2,p_{k+1}}(\Omega)$ .

If  $p_k \geq N$ , the trace theorem implies that  $\omega \in W_{loc}^{1,p}(\partial\Omega)$  for any  $p > 2$ , and then we

may repeat the above argument with arbitrarily chosen  $p_{k+1} \geq \frac{N-1}{N-2}p_k$  to infer again that  $\psi \in W^{2,p_{k+1}}(\Omega)$ .

We thus conclude that (5.11) holds, and hence (5.10) follows. By passing to the trace again, we then conclude that  $\omega \in C^{2,\alpha}(S^{N-1})$ . Consequently,  $\omega \in C^{2,\alpha}(S^{N-1}) \cap \tilde{W}_j^2 = \mathcal{X}_j$ , and thus  $\mathbb{L}_j : \mathcal{X}_j \rightarrow \mathcal{Y}_j$  is also surjective. Hence (5.5) is true.

We finally derive (5.3) from the identity

$$\partial_\lambda \Big|_{\lambda=\lambda_j} \mathbb{L}_\lambda Y_j = \partial_\lambda \Big|_{\lambda=\lambda_j} \sigma_j(\lambda) Y_j = \sigma'_j(\lambda_j) Y_j \quad (5.13)$$

and Proposition 3.5.  $\square$

**Proof of Theorem 1.1 (completed).** Recalling (5.1) and (2.11), we shall apply the Crandall-Rabinowitz bifurcation Theorem to solve the equation

$$G(\lambda, \varphi) = H(\lambda + \varphi) - H(\lambda) = \partial_{\nu_\phi} u_\phi(e_1, \cdot) - \tilde{u}'(\lambda) = 0, \quad (5.14)$$

where  $\phi = \lambda + \varphi \in \mathcal{U}$  and the function  $u_\phi \in C^{2,\alpha}(\overline{\Omega_\phi})$  is the unique solution to the Dirichlet boundary value problem

$$\begin{cases} -\Delta_{g_\phi} u_\phi = 1 & \text{in } \Omega, \\ u_\phi = 0 & \text{on } \partial\Omega, \end{cases}$$

we obtained from Lemma 2.1. Once this is done, (2.12) follows and thus we get (2.10), which is equivalent to (2.7) with  $-c = H(\lambda)$ .

To solve equation (5.14), we fix  $j \geq 2$ . Moreover, we let  $\lambda_j$  be defined as in Proposition 3.5, and we let  $\mathcal{X}_j$  be defined as in (5.4). By Proposition 5.1 and the Crandall-Rabinowitz Theorem (see [4, Theorem 1.7]), we then find  $\varepsilon_j > 0$  and a smooth curve

$$(-\varepsilon_j, \varepsilon_j) \rightarrow \mathcal{O}, \quad s \mapsto (\lambda_j(s), \varphi_s^j)$$

such that

- (i)  $G(\lambda_j(s), \varphi_s^j) = 0$  for  $s \in (-\varepsilon_j, \varepsilon_j)$ .
- (ii)  $\lambda_j(0) = \lambda_j$ .
- (iii)  $\varphi_s^j = s(Y_j + w_s^j)$  for  $s \in (-\varepsilon_j, \varepsilon_j)$  with a smooth curve

$$(-\varepsilon_j, \varepsilon_j) \rightarrow \mathcal{X}_j, \quad s \mapsto w_s^j$$

satisfying  $w_0^j = 0$ .

Since  $(\lambda_j(s), \varphi_s^j)$  is a solution to (5.14) for every  $s \in (-\varepsilon_j, \varepsilon_j)$ , the function  $u_{\phi_s^j} \in C^{2,\alpha}(\overline{\Omega})$  solves the overdetermined boundary value problem

$$\begin{cases} -\Delta_{g_{\phi_s^j}} u_{\phi_s^j} = 1 & \text{in } \Omega, \\ u_{\phi_s^j} = 0, \quad \partial_\nu u_{\phi_s^j} = \tilde{u}'(\lambda_j(s)) & \text{on } \partial\Omega, \end{cases}$$

where  $\phi_s^j = \lambda_j(s) + \varphi_s^j$ . It then follows from the reformulation procedure set up in Section 2 that the map  $s \mapsto (\lambda_j(s), \varphi_s^j)$  and the function  $u := u_{\phi_s^j} \circ \Psi_{\phi_s^j}^{-1} \circ \Upsilon^{-1} : D_{\phi_s^j} \rightarrow \mathbb{R}$  have the properties asserted in Theorem 1.1.  $\square$

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